CALCULUS BC
9.3 – Parametric Equations and Calculus

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$,

then the slope of C at the point (x, y) is given by $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$ where $\dfrac{dx}{dt} \neq 0$,

and the second derivative is given by $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left[\dfrac{dy}{dx}\right] = \dfrac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$.

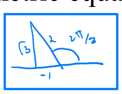
Ex. 1 (Noncalculator)

Given the parametric equations $x = t^2 + 9$ and $y = 4t^5 - 6t^3 + 3$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of t.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{20t^4 - 18t^2}{2t} = 10t^3 - 9t$$

$$\frac{dy^2}{dx^2} = \frac{30t^2 - 9}{2t} = 15t - \frac{9}{2t}$$

Ex. 2 (Noncalculator)
Given the parametric equations $x = 5\sin t$ and $y = 3\cos t$, write an equation of the tangent line to the curve at the point where $t = \dfrac{2\pi}{3}$.

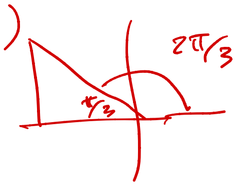


point : $\left(\frac{5\sqrt{3}}{2}, -\frac{3}{2}\right)$

$$\frac{dy}{dx} = \frac{-3\sin t}{5\cos t} = -\frac{3}{5}\tan t \text{ (slope of tangent line)}$$

$$y + \frac{3}{2} = -\frac{3}{5}(-\sqrt{3})\left(x - \frac{5\sqrt{3}}{2}\right)$$

(,)



Ex 3 (Noncalculator)
Find all points of horizontal and vertical tangency given the parametric equations

$x = 3t^2 - 6t,\ \ y = t^2 - 8t + 9.$

$x' = 6t - 6 = 6(t-1) = 0$

$t = 1$

vertical tangency @ $t = 1$, $(3 - 6,\ 1 - 8 + 9)$

$(-3, 2)$

horizontal

Earlier in the year we learned to find the arc length of a curve C given by $y = h(x)$ over the interval $x_1 \le x \le x_2$ by using the formula $s = \int_{x_1}^{x_2} \sqrt{1 + (h'(x))^2}\, dx = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$

If C is represented by the parametric equations $x = f(t)$ and $y = g(t)$ over the interval $a \le t \le b$, then

$$\int_a^b \sqrt{1 + \left[\frac{g'(t)}{f'(t)}\right]^2}\, dt \quad \left(\text{since } \frac{dy}{dx} = \frac{g'(t)}{f'(t)}\right)$$

$$\int_a^b \sqrt{\frac{(f'(t))^2}{(f'(t))^2} + \frac{(g'(t))^2}{(f'(t))^2}}\, dt \quad \int_a^b \frac{f'(t)^2 + (g'(t))^2}{[f'(t)]^2}\, dt$$

$$\int_a^b \sqrt{f'(t)^2 + (g'(t))^2}\, dt$$

Length of arc for parametric graphs is $s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$.

Note that the formula works when the curve does not intersect itself on the interval $a \le t \le b$, and the curve must be smooth.

Ex. 4 (Noncalculator)
Set up an integral expression for the arc length of the curve given by the parametric equations $x = e^{5t+2}$, $y = 3\sin(t^2)$, $0 \le t \le 4$. Do not evaluate.

$$\int_0^4 \sqrt{(5e^{5t+2})^2 + [2t(3\cos t^2)]^2}\, dt$$

Ex. Which of the following gives the length of the path described by the parametric equations $x = \sin\left(t^3\right)$ and $y = e^{5t}$ from $t = 0$ to $t = \pi$?

(A) $\int_0^{\pi} \sqrt{\sin^2\left(t^3\right) + e^{10t}}\, dt$ (B) $\int_0^{\pi} \sqrt{\cos^2\left(t^3\right) + e^{10t}}\, dt$ (C) $\int_0^{\pi} \sqrt{9t^4 \cos^2\left(t^3\right) + 25e^{10t}}\, dt$

(D) $\int_0^{\pi} \sqrt{3t^2 \cos\left(t^3\right) + 5e^{5t}}\, dt$ (E) $\int_0^{\pi} \sqrt{\cos^2\left(t^3\right) + e^{10t}}\, dt$

$$\left[\frac{dx}{dt}\right]^2 = 2\left(\sin^2 t^3\right)\left(3t \cos t^3\right)$$

Ex. A curve C is defined by the parametric equations $x = t^2 - 4t + 1$ and $y = t^3$. Which of the following is an equation of the line tangent to the graph of C at the point $(-3, 8)$?

(A) $x = -3$ (B) $x = 2$ (C) $y = 8$ (D) $y = -\frac{27}{10}(x+3) + 8$ (E) $y = 12(x+3) + 8$

$y = 8 = t^3 \quad t = 2$

$x = -3 = (2)^2 - 4(2) + 1 = 4 - 7 = -3$ ✓

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \left.\frac{3t^2}{2t-4}\right|_{t=2} = \frac{3(2)^2}{2(2)-4} = \text{vertical tangent}$$

Ex. The length of the path described by the parametric equations $x = \frac{1}{3}t^3$ and $y = \frac{1}{2}t^2$, where $0 \le t \le 1$, is given by

(A) $\int_0^1 \sqrt{t^2 + 1}\, dt$ (B) $\int_0^1 \sqrt{t^2 + t}\, dt$ (C) $\int_0^1 \sqrt{t^4 + t^2}\, dt$ (D) $\frac{1}{2}\int_0^1 \sqrt{4 + t^4}\, dt$ (E) $\frac{1}{6}\int_0^1 t^2\sqrt{4t^2 + 9}\, dt$

$\left(\frac{dx}{dt}\right)^2 = \left(t^2\right)^2 = t^4$

$\left(\frac{dy}{dt}\right)^2 = (t)^2 = t^2$

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\int_0^1 \sqrt{t^4 + t^2}$$

Ex. A particle moves in the xy-plane so that its position for $t \geq 0$ is given by the parametric equations $x = \ln(t+1)$ and $y = kt^2$, where k is a positive constant. The line tangent to the particle's path at the point where $t = 3$ has slope 8. What is the value of k?

$$8 = \left.\frac{dy}{dx}\right|_{t=3} = \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=3} = \left.\frac{2kt}{\frac{1}{t+1}}\right|_{t=3} = 2kt(t+1)\Big|_{t=3}$$

$$8 = (3)2k(4)$$

$$= \quad 8 = 24k$$

$$k = 1/3$$

Ex. A curve is defined by the parametric equations $x(t) = 3e^{2t}$ and $y(t) = e^{3t} - 1$. What is $\frac{d^2y}{dx^2}$ in terms of t?

$$\frac{dy}{dt} = \frac{3e^{3t}}{6e^{2t}} = \frac{3}{6}e^{3t-2t} = \frac{1}{2}e^{t}$$

$$\frac{dy^2}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dt}\right)/dt}{\frac{dx}{dt}} = \frac{\frac{1}{2}e^{t}}{6e^{2t}} = \frac{1}{12e^{t}}$$